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THY-ANGLE IN AN *n*-NORMED SPACE

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Abstract

In this paper, we introduce the notion of angle and Thy-angle in an n-normed space and prove some fundamental properties and a characterization theorem for Thy-angle in an n-normed space.

1. Introduction

The notion of Thy-angle in normed space has been introduced in [15, 16]. The study on an n-normed space, which is a generalization of normed space, has been done by mathematicians as outlined in [2-11]. By considering those, a notion of Thy-angle in an n-normed space, a generalization of Thy-angle, is already constructed in [15].

On the other hand, in 2008, Thillaigovindan et al. [14] have constructed a notion of angle between two vectors in an n-normed space. However, there are some gaps in Definition 3.1 [14]. The gaps are: (i) if $\{x_2, \ldots, x_n\}$ is linearly dependent, then $\|x, x_2, \ldots, x_n\|$ or $\|y, x_2, \ldots, x_n\|$ is zero, so

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$\theta(x, y; x_2, ..., x_n)$ in [14] is not defined on X; (ii) if x or y is in $span\{x_2, ..., x_n\}$, then $\theta(x, y; x_2, ..., x_n)$ is not defined on X. By the existence of gaps, a notion of angle between two vectors in an n-normed space is developed which is an improvement of Definition 3.1 [14].

In this paper, the notion of Thy-angle in an n-normed space along with the new notion of angle between two vectors in an n-normed space, is used for proving a characterization theorem for Thy-angle in an n-normed space.

2. Preliminaries

This section provides a collection of basic definitions and results needed for the development of the new concepts.

Definition 2.1 [8]**.** Let n be a natural number $(n \in \mathbb{N})$ and X be a real vector space having $\dim(X) \geq n$. A real valued function $\| \cdot, ..., \cdot \|$ on X^n is said to be an *n-norm* on X if for any $x_1, x_2, ..., x_n \in X$, the following four properties hold:

(i) $\| x_1, ..., x_n \| = 0$ if and only if $\{x_1, x_2, ..., x_n\}$ is linear dependent on X,

(ii) $\| x_1, ..., x_n \|$ is invariant under any permutation,

(iii) $\| \alpha x_1, ..., x_n \| = | \alpha | \| x_1, ..., x_n \|$ for any $\alpha \in \mathbb{R}$,

(iv) $\| x + y, ..., x_n \| \leq \| x, ..., x_n \| + \| y, ..., x_n \|$.

The pair $(X, \| \cdot, ..., \cdot \|)$ is called an *n-normed space* on X.

Definition 2.2 [1]**.** Let $n \in \mathbb{N}$ and X be a real vector space with $\dim(X) \geq n$. A real valued function $\langle \cdot, \cdot | \cdot, ..., \cdot \rangle$ on X^{n+1} is said to be an *n-inner product* on X if for any $x_1, x_2, ..., x_n \in X$ satisfying the following conditions:

(i) $\langle x, x | x_2, x_3, ..., x_n \rangle \geq 0$,

(ii) $\langle x, x | x_2, x_3, ..., x_n \rangle = 0$ if and only if $\{x, x_2, ..., x_n\}$ is linear dependent,

(iii) $\langle x, y | x_2, x_3, ..., x_n \rangle = \langle y, x | x_2, x_3, ..., x_n \rangle$,

(iv) $\langle x, y | x_2, x_3, ..., x_n \rangle$ is invariant under any permutation of $x_2, ..., x_n$,

(v) $\langle x, x | x_2, x_3, ..., x_n \rangle = \langle x_2, x_2 | x, x_3, ..., x_n \rangle$,

(vi) $\langle \alpha x, x | x_2, x_3, ..., x_n \rangle = \alpha \langle x, x | x_2, x_3, ..., x_n \rangle$ for any $\alpha \in \mathbb{R}$,

(vii) $\langle x + x', y | x_2, x_3, ..., x_n \rangle = \langle x, y | x_2, x_3, ..., x_n \rangle + \langle x', y | x_2, x_3, ..., x_n \rangle$.

The pair $(X, \langle \cdot, \cdot | \cdot, ..., \cdot \rangle)$ is called an *n-inner product space* on X.

Theorem 2.3 [1]**.** *If an n-inner product space on X is given, then*

(i) $\| x_1, x_2, ..., x_n \| = \sqrt{\langle x_1, x_1 | x_2, x_3, ..., x_n \rangle}$ *defines an n-norm on X*,

(ii) *the extension of Cauchy-Buniakowski inequality*:

$$\langle x, y | x_2, x_3, ..., x_n \rangle \leq \sqrt{\langle x, x | x_2, x_3, ..., x_n \rangle} \cdot \sqrt{\langle y, y | x_2, x_3, ..., x_n \rangle}$$

also holds.

Theorem 2.4 [12]**.** *If an n-inner product space on X is given, then* $\| x_1, x_2, ..., x_n \| = \sqrt{\langle x_1, x_1 | x_2, x_3, ..., x_n \rangle}$ *defines an n-norm on X for which*

$$\langle x, y | x_2, x_3, ..., x_n \rangle = \frac{1}{4}\{\| x + y, x_2, ..., x_n \|^2 - \| x - y, x_2, ..., x_n \|^2\},$$

$$\langle x, y | x_2, x_3, ..., x_n \rangle = \frac{1}{2}\{\| x, x_2, ..., x_n \|^2 + \| y, x_2, ..., x_n \|^2 - \| x - y, x_2, ..., x_n \|^2\}$$

and

$$\| x + y, x_2, ..., x_n \|^2 + \| x - y, x_2, ..., x_n \|^2$$

$$= 2\{\| x, x_2, ..., x_n \|^2 + \| y, x_2, ..., x_n \|^2\}.$$

Definition 2.5 [15]**.** Let $(X, \| \cdot \|)$ be a normed space and $x, y \in X$. A Thy-angle between *x* and *y*, denoted by $A_{Thy}(x, y)$, is defined as

$$A_{Thy}(x, y) = \arccos\left[\frac{1}{4}\left(\left\| \frac{x}{\| x \|} + \frac{y}{\| y \|} \right\|^2 - \left\| \frac{x}{\| x \|} - \frac{y}{\| y \|} \right\|^2\right)\right].$$

Theorem 2.6 [15]**.** $A_{Thy}(x, y)$ *lies between* 0 *and* π.

3. Discussion

3.1. Thy-angle in an *n*-normed space

In this section, we will use *X* is a real vector space with $\dim(X) \geq n$.

Definition 3.1.1. Let $(X, \| \cdot, ..., \cdot \|)$ be an *n*-normed space and $x, y, x_1 x_2, x_3, ..., x_n \in X$, $\{x_1, x_2, x_3, ..., x_n\}$ be a linearly independent set. A Thy-angle between *x* and *y* with respect to $x_1, x_2, x_3, ..., x_n$, denoted by $A_{Thy}(x, y; x_1, ..., x_n)$, is defined as

$$A_{Thy}(x, y; x_1, ..., x_n)$$

$$= \arccos\left[\frac{1}{4} \sum_{\{i_2, ..., i_n\} \subset \{1, ..., n\}} \left(\left\| \frac{x}{\varphi} + \frac{y}{\psi}, x_{i_2}, ..., x_{i_n} \right\|^2 \right.\right.$$

$$\left.\left. - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, ..., x_{i_n} \right\|^2\right)\right]$$

with

$$\varphi = \left(\sum_{\{i_2, ..., i_n\} \subset \{1, ..., n\}} \| x, x_{i_2}, ..., x_{i_n} \|^2 \right)^{\frac{1}{2}}$$

and

$$\psi = \left(\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \| y, x_{i_2}, \ldots, x_{i_n} \|^2\right)^{\frac{1}{2}}.$$

Theorem 3.1.2. *Let* $(X, \|\cdot, \ldots, \cdot\|)$ *be an* n*-normed space and* $x, y, x_1, x_2, x_3, \ldots, x_n \in X$. *If* $\{x_1, x_2, \ldots, x_n\}$ *is a linearly independent set, then*

(i) $A_{Thy}(x, y; x_1, \ldots, x_n) = A_{Thy}(y, x; x_1, \ldots, x_n)$.

(ii) $A_{Thy}(\alpha x, \beta y; x_1, \ldots, x_n) = A_{Thy}(x, y; x_1, \ldots, x_n)$, *for* $\alpha, \beta \in \mathbb{R}$ *with* $\beta > 0$.

(iii) $A_{Thy}(\alpha x, \beta y; x_1, \ldots, x_n) = \pi - A_{Thy}(x, y; x_1, \ldots, x_n)$, *for* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta < 0$.

(iv) $A_{Thy}(x, y; x_1, \ldots, x_n) = 0$ *if and only if x and y are two vectors of the same direction*.

(v) $A_{Thy}(x, y; x_1, \ldots, x_n) = \pi$ *if and only if x and y are in opposite direction*.

(vi) *If* $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ *is an* n*-inner product space, then*

$$A_{Thy}(x, y; x_1, \ldots, x_n) = arccos\left[\frac{\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \langle x, y | x_{i_2}, \ldots, x_{i_n} \rangle}{\varphi \cdot \psi}\right]$$

with

$$\varphi = \left(\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \| x, x_{i_2}, \ldots, x_{i_n} \|^2\right)^{\frac{1}{2}}$$

and

$$\psi = \left(\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \| y, x_{i_2}, \ldots, x_{i_n} \|^2\right)^{\frac{1}{2}}.$$

(vii) *If* $\varphi = \psi$, *then*

$$4\varphi^2 \cos A_{Thy}(x, y; x_1, .., x_n)$$

$$= \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} (\| x + y, x_{i_2}, \dots, x_{i_n} \|^2 - \| x - y, x_{i_2}, \dots, x_{i_n} \|^2)$$

with

$$\varphi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \| x, x_{i_2}, \dots, x_{i_n} \|^2 \right)^{\frac{1}{2}}$$

and

$$\psi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \| y, x_{i_2}, \dots, x_{i_n} \|^2 \right)^{\frac{1}{2}}.$$

Proof. For (i) to (v), it follows immediately from Definition 3.1.1.

Since $\{x_1, x_2, \dots, x_n\}$ is a linearly independent set, by applying Theorem 2.4, we have

$$\left\langle \frac{x}{\varphi}, \frac{y}{\psi} \middle| x_{i_2}, \dots, x_{i_n} \right\rangle = \frac{1}{4} \left\{ \left\| \frac{x}{\varphi} + \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right\}.$$

So,

$$\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\langle \frac{x}{\varphi}, \frac{y}{\psi} \middle| x_{i_2}, \dots, x_{i_n} \right\rangle = \cos A_{Thy}(x, y; x_1, \dots, x_n).$$

This proves (vi).

To prove (vii) we proceed as follows.

Since $\varphi = \psi$,

$$\frac{1}{4}\sum_{\{i_2, \ldots, i_n\}\subset\{1, \ldots, n\}}\left(\left\|\frac{x}{\varphi}+\frac{y}{\psi}, x_{i_2}, \ldots, x_{i_n}\right\|^2-\left\|\frac{x}{\varphi}-\frac{y}{\psi}, x_{i_2}, \ldots, x_{i_n}\right\|^2\right)$$

$$= \frac{1}{4\varphi^2}\sum_{\{i_2, \ldots, i_n\}\subset\{1, \ldots, n\}}(\| x+y, x_{i_2}, \ldots, x_{i_n}\|^2 - \| x-y, x_{i_2}, \ldots, x_{i_n}\|^2).$$

Consequently,

$$4\varphi^2 \cos A_{Thy}(x, y; x_1, \ldots, x_n)$$

$$= \sum_{\{i_2, \ldots, i_n\}\subset\{1, \ldots, n\}}(\| x+y, x_{i_2}, \ldots, x_{i_n}\|^2 - \| x-y, x_{i_2}, \ldots, x_{i_n}\|^2). \qquad \square$$

Theorem 3.1.3. *Let* $x, y, x_1, x_2, \ldots, x_n \in X$. *If* $\{x_1, x_2, \ldots, x_n\}$ *is a linearly independent set, then for any real number* $t \neq 0$, *we have*

$$\lim_{t\to\infty}\left(\frac{tx+y}{\left(\sum_{\{i_2, \ldots, i_n\}\subset\{1, \ldots, n\}}\| tx+y, x_{i_2}, \ldots, x_{i_n}\|^2\right)^{\frac{1}{2}}}\right)$$

$$= \frac{x}{\left(\sum_{\{i_2, \ldots, i_n\}\subset\{1, \ldots, n\}}\| x, x_{i_2}, \ldots, x_{i_n}\|^2\right)^{\frac{1}{2}}}$$

and

$$\lim_{t\to-\infty}\left(\frac{tx+y}{\left(\sum_{\{i_2, \ldots, i_n\}\subset\{1, \ldots, n\}}\| tx+y, x_{i_2}, \ldots, x_{i_n}\|^2\right)^{\frac{1}{2}}}\right)$$

$$= \frac{-x}{\left(\sum_{\{i_2, \ldots, i_n\}\subset\{1, \ldots, n\}}\| x, x_{i_2}, \ldots, x_{i_n}\|^2\right)^{\frac{1}{2}}}.$$

Proof. Let $|t|$ be sufficiently large with $tx + y \neq 0$.

If $t > 0$, then

$$\lim_{t\to\infty}\left(\frac{tx + y}{\left(\sum_{\{i_2, \dots, i_n\}\subset\{1, \dots, n\}}\| tx + y, x_{i_2}, \dots, x_{i_n}\|^2\right)^{\frac{1}{2}}}\right)$$

$$= \lim_{t\to\infty}\left(\frac{x + y/t}{\left(\sum_{\{i_2, \dots, i_n\}\subset\{1, \dots, n\}}\| x + y/t, x_{i_2}, \dots, x_{i_n}\|^2\right)^{\frac{1}{2}}}\right)$$

$$= \frac{x}{\left(\sum_{\{i_2, \dots, i_n\}\subset\{1, \dots, n\}}\| x, x_{i_2}, \dots, x_{i_n}\|^2\right)^{\frac{1}{2}}}.$$

If $t < 0$, then

$$\frac{tx + y}{\left(\sum_{\{i_2, \dots, i_n\}\subset\{1, \dots, n\}}\| tx + y, x_{i_2}, \dots, x_{i_n}\|^2\right)^{\frac{1}{2}}}$$

$$= \frac{-x + y/k}{\left(\sum_{\{i_2, \dots, i_n\}\subset\{1, \dots, n\}}\| -x + y/k, x_{i_2}, \dots, x_{i_n}\|^2\right)^{\frac{1}{2}}},$$

where $t = -k$, $k > 0$ and

$$\lim_{t\to-\infty}\left(\frac{tx + y}{\left(\sum_{\{i_2, \dots, i_n\}\subset\{1, \dots, n\}}\| tx + y, x_{i_2}, \dots, x_{i_n}\|^2\right)^{\frac{1}{2}}}\right)$$

$$= -\frac{x}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \| x, x_{i_2}, \dots, x_{i_n} \|^2\right)^{\frac{1}{2}}}. \qquad \square$$

Theorem 3.1.4. *$A_{Thy}(x, y, x_1, \dots, x_n)$ lies between* 0 *and* π.

Proof. Let

$$\varphi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \| x, x_{i_2}, \dots, x_{i_n} \|^2\right)^{\frac{1}{2}}$$

and

$$\psi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \| y, x_{i_2}, \dots, x_{i_n} \|^2\right)^{\frac{1}{2}}.$$

Since

$$\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\| \frac{x}{\varphi} + \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2$$

$$\leq \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\varphi}, x_{i_2}, \dots, x_{i_n} \right\| + \left\| \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\| \right)^2$$

$$\leq \left[\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\| \frac{x}{\varphi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right)^{\frac{1}{2}} \right.$$

$$\left. + \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\| \frac{y}{\varphi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right)^{\frac{1}{2}} \right]^2 = 4$$

and

$$\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right)$$

$$\leq \left[\left(\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \left\| \frac{x}{\varphi}, x_{i_2}, \ldots, x_{i_n} \right\|^2 \right)^{\frac{1}{2}} \right.$$

$$\left. + \left(\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \left\| \frac{y}{\varphi}, x_{i_2}, \ldots, x_{i_n} \right\|^2 \right) \right]^2 = 4,$$

we have

$$-4 \leq \sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \left(\left\| \frac{x}{\varphi} + \frac{y}{\psi}, x_{i_2}, \ldots, x_{i_n} \right\|^2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \ldots, x_{i_n} \right\|^2 \right)$$

$$\leq 4.$$

Therefore $A_{Thy}(x, y, x_1, \ldots, x_n)$ lies between 0 and π. □

Theorem 3.1.5. *Let* $x, y, x_1, x_2, x_3, \ldots, x_n \in X$. *If* $\{x_1, x_2, \ldots, x_n\}$ *is a linearly independent set, then for each* $\alpha \in (0, \pi)$, *there exists a real number a such that*

$$A_{Thy}(x, ax + y; x_1, \ldots, x_n) = \alpha.$$

Proof. By using Theorem 3.1.5, it is sufficient to see that for any $k \in (-1, 1)$ there exists a number *a* such that

$$\frac{1}{4} \sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \left(\left\| \frac{x}{\varphi} + \frac{y}{\psi}, x_{i_2}, \ldots, x_{i_n} \right\|^2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \ldots, x_{i_n} \right\|^2 \right) = k.$$

By using Theorem 3.1.4, we have

$$\lim_{t \to \infty} \left(\frac{1}{4} \sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \left(\left\| \frac{x}{\varphi} + \frac{tx + y}{\omega}, x_{i_2}, \ldots, x_{i_n} \right\|^2 \right. \right.$$

$$\left. \left. - \left\| \frac{x}{\varphi} - \frac{tx + y}{\omega}, x_{i_2}, \ldots, x_{i_n} \right\|^2 \right) \right) = 1$$

and

$$\lim_{t \to -\infty}\left(\frac{1}{4}\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}}\left(\left\| \frac{x}{\varphi} + \frac{tx + y}{\omega}, x_{i_2}, \ldots, x_{i_n} \right\|^2 - \left\| \frac{x}{\varphi} - \frac{tx + y}{\omega}, x_{i_2}, \ldots, x_{i_n} \right\|^2\right)\right) = -1$$

with

$$\varphi = \left(\sum\nolimits_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \| x, x_{i_2}, \ldots, x_{i_n} \|^2\right)^{\frac{1}{2}},$$

$$\psi = \left(\sum\nolimits_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \| y, x_{i_2}, \ldots, x_{i_n} \|^2\right)^{\frac{1}{2}},$$

and

$$\omega = \left(\sum\nolimits_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \| tx + y, x_{i_2}, \ldots, x_{i_n} \|^2\right)^{\frac{1}{2}}.$$

Since

$$g(t) = \frac{1}{4}\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}}\left(\left\| \frac{x}{\varphi} + \frac{tx + y}{\psi}, x_{i_2}, \ldots, x_{i_n} \right\|^2 - \left\| \frac{x}{\varphi} - \frac{tx + y}{\psi}, x_{i_2}, \ldots, x_{i_n} \right\|^2\right)$$

is a continuous function at t, the result follows. □

3.2. The angle between two vectors

The following definition is a revised version of Definition 3.1 [14].

Definition 3.2.1. Let $(X, \|\cdot, ..., \cdot\|)$ be an n-normed space and $x, y, x_1 x_2, x_3, ..., x_n \in X$, $\{x_1, x_2, x_3, ..., x_n\}$ be a linearly independent set. The angle between x and y with respect to $x_1, x_2, x_3, ..., x_n$, denoted by $A_*(x, y; x_1, ..., x_n)$, is defined as

$$A_*(x, y; x_1, ..., x_n)$$

$$= \arccos\left[\frac{1}{2}\sum_{\{i_2, ..., i_n\}\subset\{1, ..., n\}}\left(2 - \left\|\frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, ..., x_{i_n}\right\|^2\right)\right]$$

with

$$\varphi = \left(\sum_{\{i_2, ..., i_n\}\subset\{1, ..., n\}}\|x, x_{i_2}, ..., x_{i_n}\|^2\right)^{\frac{1}{2}}$$

and

$$\psi = \left(\sum\nolimits_{\{i_2, ..., i_n\}\subset\{1, ..., n\}}\|y, x_{i_2}, ..., x_{i_n}\|^2\right)^{\frac{1}{2}}.$$

The relation between A_{Thy} and A_* which expressed in the following theorem, is called as a *characterization theorem* for Thy-angle.

Theorem 3.2.2 (Characterization theorem)**.** *Let* $(X, \|\cdot, ..., \cdot\|)$ *be an n-normed space. The pair* $(X, \langle\cdot, \cdot|\cdot, ..., \cdot\rangle)$ *is an n-inner product space if and only if*

$$A_{Thy}(x, y; x_1, ..., x_n) = A_*(x, y; x_1, ..., x_n)$$

for any $x, y, x_1, x_2, ..., x_n \in X$, $\{x_1, x_2, ..., x_n\}$ *is a linearly independent set.*

Proof. By using (vi) of Theorem 3.1.2, we have

$$A_{Thy}(x, y; x_2, \ldots, x_n) = \arccos\left[\frac{\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \langle x, y \mid x_{i_2}, \ldots, x_{i_n} \rangle}{\varphi \cdot \psi}\right].$$

We also have

$$A_{**}(x, y; x_2, \ldots, x_n) = \arccos\left[\frac{\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \langle x, y \mid x_{i_2}, \ldots, x_{i_n} \rangle}{\varphi \cdot \psi}\right]$$

with

$$\varphi = \left(\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \| x, x_{i_2}, \ldots, x_{i_n} \|^2\right)^{\frac{1}{2}}$$

and

$$\psi = \left(\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \| y, x_{i_2}, \ldots, x_{i_n} \|^2\right)^{\frac{1}{2}}.$$

By applying Theorem 2.4 to $\frac{x}{\varphi}$ and $\frac{y}{\psi}$, for any $x, y, x_1, \ldots, x_n \in X$, $\{x_1, x_2, \ldots, x_n\}$ is a linearly independent set, we have

$$\sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \left\langle \frac{x}{\varphi}, \frac{y}{\psi} \middle| x_{i_2}, \ldots, x_{i_n} \right\rangle$$

$$= \sum_{\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}} \frac{1}{2}\left\{\left\| \frac{x}{\varphi}, x_{i_2}, \ldots, x_{i_n} \right\|^2 + \left\| \frac{y}{\psi}, x_{i_2}, \ldots, x_{i_n} \right\|^2\right.$$

$$\left. - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \ldots, x_{i_n} \right\|^2\right\}$$

$$= \sum_{\{i_2, ..., i_n\} \subset \{1, ..., n\}} \frac{1}{2}\left\{2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, ..., x_{i_n} \right\|^2\right\}$$

$$= \cos A_*(x, y; x_1, ..., x_n).$$

So, $A_{Thy}(x, y; x_1, ..., x_n) = A_*(x, y; x_1, ..., x_n)$.

Conversely, let

$$A_{Thy}(x, y; x_1, ..., x_n) = A_*(x, y; x_1, ..., x_n)$$

for any $x, y, x_1, x_2, ..., x_n \in X$, with $\{x_2, x_3, ..., x_n\}$ be a linear independent set. By applying Theorem 2.4, it is enough to see that parallelogram law holds for $x, y, x_1, x_2, ..., x_n \in X$, $\{x_1, x_2, ..., x_n\}$ is a linear independent set.

If we take $\varphi = \psi = 1$, then

$$\cos A_{Thy}(x, y; x_1, ..., x_n)$$

$$= \frac{1}{4} \sum_{\{i_2, ..., i_n\} \subset \{1, ..., n\}} (\| x + y, x_{i_2}, ..., x_{i_n} \|^2 - \| x - y, x_{i_2}, ..., x_{i_n} \|^2),$$

and

$$\cos A_*(x, y; x_2, ..., x_n) = \sum_{\{i_2, ..., i_n\} \subset \{1, ..., n\}} \frac{1}{2}\left\{2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, ..., x_{i_n} \right\|^2\right\}.$$

Therefore,

$$\frac{1}{4} \sum_{\{i_2, ..., i_n\} \subset \{1, ..., n\}} (\| x + y, x_{i_2}, ..., x_{i_n} \|^2 - \| x - y, x_{i_2}, ..., x_{i_n} \|^2)$$

$$= \sum_{\{i_2, ..., i_n\} \subset \{1, ..., n\}} \frac{1}{2}\left\{2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, ..., x_{i_n} \right\|^2\right\}.$$

Since $x, y, x_1, x_2, ..., x_n \in X$ are arbitrary,

$$\| x - y, x_2, ..., x_n \|^2 - \| x + y, x_2, ..., x_n \|^2 = 4 - 2\| x - y, x_2, ..., x_n \|^2.$$

Hence

$$\| x - y, x_2, ..., x_n \|^2 + \| x + y, x_2, ..., x_n \|^2 = 4.$$

It shows that $(X, \langle \cdot, \cdot | \cdot, ..., \cdot \rangle)$ is an n-inner product space. □

References

[1] Y. J. Cho, M. Matic and J. Pacarie, Inequalities of Hlawka's type in n-inner product spaces, Comm. Korean Math. Soc. 17(4) (2002), 583-592.

[2] Y. J. Cho, P. C. S. Lin, S. S. Kim and A. Misiak, Theory of 2-inner Product Spaces, Nova Science Publisher, Inc., Huntington, NY, 2001.

[3] C. Diminnie, S. Gähler and A. White, 2-inner product spaces, Demonstratio Math. 6 (1973), 525-536.

[4] C. Diminnie, S. Gähler and A. White, 2-inner product spaces. II, Demonstratio Math. 10(1) (1977), 169-188.

[5] S. Gähler, Lineare 2-normietre Raume, Math. Nachr. 28 (1965), 1-43.

[6] S. Gähler, Untersuchungen uber verallgemeinerte m-metrisch Raume. I, Math. Nachr. 40 (1969), 165-189.

[7] S. Gähler, Untersuchungen uber verallgemeinerte m-metrische Raume. II, Math. Nachr. 40 (1969), 229-264.

[8] H. Gunawan and Mashadi, On n-normed spaces, Int. J. Math. Sci. 27 (2001), 631-639.

[9] H. Gunawan, Mashadi, S. Gemawati, Nursupiamin and I. Sihwaningrum, Orthogonality in 2-normed spaces revisited, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. 17 (2006), 76-83.

[10] H. Mazaheri and S. G. Nezhad, Some results on b-orthogonality in 2-normed linear spaces, Int. J. Math. Anal. 1 (2007), 681-687.

[11] A. Misiak, n-inner product spaces, Math. Nachr. 140 (1989), 299-319.

[12] A. Misiak, Orthogonality and orthonormality in n-inner product spaces, Math. Nachr. 143 (1989), 249-261.

[13] B. Surender Reddy, Intuitionistic fuzzy 2-norm, Int. J. Math. Anal. (Ruse) 5(13-16) (2011), 651-659.

[14] N. Thillaigovindan, S. Vijayabalaji and S. Anita Shanthi, Angles in *n*-normed linear space, Proceedings of National Seminar on Mathematical Methods for Research, A. V. C. College, 2008, pp. 61-68.

[15] V. Thürey, Angles and polar coordinates in real normed spaces, arXiv:0902.2731v2.

[16] V. Thürey, Angles and a classification of normed spaces, Ann. Funct. Anal. 4(1) (2013), 114-137.